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399th solution of the Ising model

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Abstract. We show that the nearest-neighbour correlations of the honeycomb, triangular and square Ising models can be obtained by using *only* the star-triangle relations and simple assumptions concerning the thermodynamic limit and differentiability. This gives the internal energy, and hence the free energy and specific heat.

1. Introduction

Since the original solution of the two-dimensional Ising model by Onsager (1944), many alternative derivations have been given. Onsager diagonalised the transfer matrix by looking for irreducible representations of a related matrix algebra; Kaufman (1949) simplified this derivation by using spinor operators; Schultz *et al* (1964), and Thompson (1965), further simplified it by using fermion operators.

Kac and Ward (1952) used combinatorial arguments to write the partition function as a determinant. This method was refined by Potts and Ward (1955).

Hurst and Green (1960), and Kasteleyn (1963) also used combinatorial arguments, but this time to write the partition function as a Pfaffian. Another combinatorial solution was obtained by Vdovichenko (1965), and is given by Landau and Lifschitz (1968). A full account of the Pfaffian method and the properties of the Ising model is given by McCoy and Wu (1973).

More recently, the Ising model has been solved by making use of the commutation properties of the transfer matrix (Baxter 1972, Stephen and Mittag 1972).

In this paper we give a new derivation of the solution. We believe it to be the simplest yet, since it involves *only* a local property of the Ising model, namely the star-triangle relation, together with some straightforward assumptions concerning the thermodynamic limit and differentiability. It does *not* involve transfer matrices or combinatorial arguments.

The derivation rests on two ideas, which have previously been used by the two of us, respectively (Baxter 1978, Enting 1977).

The first idea was developed for the eight-vertex model. Its specialisation to the Ising model is given in § 2. By repeated use of the star-triangle transformation it is shown that each nearest-, or next-nearest-, neighbour correlation on a large anisotropic honeycomb lattice is the same as a nearest–neighbour correlation on a related square lattice. This means that each is a function of only two parameters (the square lattice interaction coefficients), instead of three.

The second idea is used in § 3. By considering a single star, a linear relation is obtained between the first and second-neighbour correlations of the honeycomb lattice.

These two results are combined in § 4 to give a functional equation for these correlations. The general solution of this equation is obtained in § 5. It still contains a single unknown function of one variable. This function is obtained in §§ 6 and 7 by using the rotation symmetry of the free energy.

The first idea has been implicit in the literature for some years. It is connected with the commutation properties of the transfer matrix. Onsager (1971) was well aware of the connection between these and the star-triangle relation. What we do believe to be new is the simple derivation of the first result and the wedding of it to the second to obtain the correlations.

Once these correlations are known as functions of the interaction coefficients, the free energy can of course be obtained by integration. Thus in one fell swoop we have solved the honeycomb, triangular and square Ising models.

These ideas can probably be extended to the eight-vertex model. It is intended to attempt this, using the Ashkin–Teller formulation (Wu 1977). Indeed, they probably apply to any model that satisfies a star-triangle relation, for instance the critical Potts model (Baxter *et al* 1978).

2. The honeycomb lattice correlations as square lattice ones

Consider an anisotropic Ising model on the honeycomb lattice. At each site i there is a spin σ_i, with values +1 or −1. The interaction energy between adjacent spins σ_i, σ_l is $-k_B T L_r \sigma_i \sigma_l$, where k_B is Boltzmann's constant, T the temperature, and r takes the values 1, 2, 3 according to the direction in which edge (i, l) lies, as indicated in figure 1. Thus there are three interaction coefficients: L_1, L_2, L_3.

The star-triangle relation is well-known (Wannier 1945, Houtappel, 1950): by summing over the spin at site l one can convert the star $i\ j\ k\ l$ (full lines) to the triangle $i\ j\ k$ (dotted lines), with interaction coefficients K_1, K_2, K_3 on the dotted edges.

The K_i are related to the L_i by

$$\exp(2K_1+2K_2)=\frac{\cosh(L_1+L_2+L_3)}{\cosh(L_1+L_2-L_3)} \tag{1}$$

and two other equations obtained by permuting the suffixes 1, 2, 3. If L_1, L_2, L_3 are all

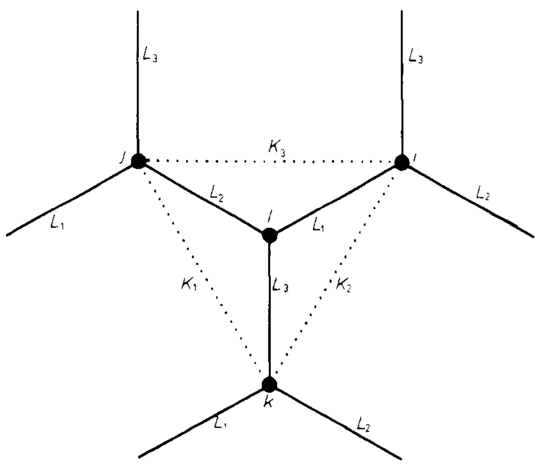

Figure 1. A star on the honeycomb lattice and its associated (dotted) triangle. The interaction coefficients for the various edges are shown.

real (positive), then so are K_1, K_2, K_3. If K_1, K_2, K_3 are positive, then L_1, L_2, L_3 can all be chosen positive.

Conversely, one can convert a triangle, with coefficients K_1, K_2, K_3, to a star, with coefficients L_1, L_2, L_3.

There are two sorts of star on the honeycomb lattice: down-pointing ones (e.g. $i\ j\ k\ l$), and up-pointing ones (e.g. the one with centre site i). By applying the star-to-triangle transformation to all down-pointing (or all up-pointing) stars, one converts the honeycomb Ising model to a triangular one with interaction coefficients K_1, K_2, K_3.

Now let us do something slightly different. Consider the honeycomb lattice shown in figure 2(a) by full lines. Let R be the central row of vertical edges, containing sites i, j, m, n. Suppose the lattice to be wound on a vertical cylinder, so the right side is joined to the left. Proceed as follows.

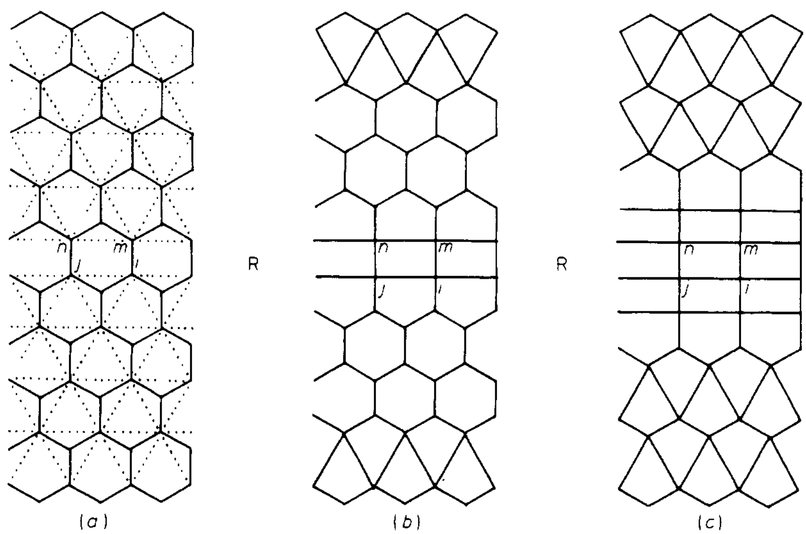

Figure 2. The effect of repeated star-to-triangle and triangle-to-star transformations on the honeycomb lattice.

(i) Perform a star-to-triangle transformation by summing over the centre spins of all up-pointing three-edge stars above R, and of all down-pointing three-edge stars below R.

This leaves the spins at sites i, j, m, n unaffected. It converts the model to one on the lattice consisting of the dotted lines in figure 2(a), the full vertical lines in row R, and the full non-vertical lines at the top and bottom boundaries. Parallel edges have the same interaction coefficient, being the same as that of the edge to which they are parallel in figure 1.

This new mixed lattice contains triangular regions above and below R. The next step is

(ii) Perform a triangle-to-star transformation on down-pointing triangles above R, up-pointing triangles below R.

This continues to leave unchanged the spins at sites i, j, m, n, but takes the model to one on the lattice shown in figure 2(b). Note that it contains a central square-lattice band including R. There are honeycomb regions above and below this band.

Now repeat steps (i) and (ii), and continue until there are no appropriate three-edge stars or triangles to transform. The result is the lattice shown in figure 2(*c*). It contains a central square lattice region, with interaction coefficient L_3 for all vertical edges, K_3 for all horizontal ones. Above and below this are regions with non-square quadrilateral faces and coefficients L_1, L_2, K_1, K_2 for the various edges.

In the thermodynamic limit the original honeycomb lattice is large, and so is the square-lattice region of figure 2(*c*). Although the kite-shaped regions beyond this are also large, they affect the square region only via its boundary spins. They are therefore equivalent to some special boundary condition on the square region. Since L_1, L_2, K_1, K_2 are all real, this boundary condition only introduces positive weights, and in the thermodynamic limit it cannot affect even-spin correlations deep within the square region.

The two-spin correlations between σ_i, σ_l, σ_m, σ_n must therefore be those of the square lattice. In particular, it must be true that

$$\langle\sigma_i\ \sigma_l\rangle = g(K_3, L_3), \tag{2a}$$

$$\langle\sigma_i\ \sigma_m\rangle = g(L_3, K_3) \tag{2b}$$

where $g(K, L)$ is the horizontal nearest-neighbour correlation of a square lattice with interaction coefficients K and L on horizontal and vertical edges, respectively.

Thus we have reduced these second- and first-neighbour correlations of the honeycomb lattice from functions of the three variables L_1, L_2, L_3 to functions of the two variables K_3, L_3.

3. Local relation between honeycomb first and second-neighbour correlations

Again consider the star shown in figure 1. Let $P(\alpha, \beta, \gamma)$ be the probability that the spins at sites i, j, k have values α, β, γ (this probability is obtained in the usual way by summing $Z^{-1}\exp(-\mathcal{H}/k_BT)$ over all spins other than those at i, j, k). Let $P(\alpha, \beta, \gamma, \delta)$ be the corresponding probability for sites i, j, k, l. Then

$$P(\alpha, \beta, \gamma, \delta) = P(\delta|\alpha, \beta, \gamma)P(\alpha, \beta, \gamma) \tag{3}$$

where $P(\delta|\alpha, \beta, \gamma)$ is the probability that the spin at l has value δ, given that the spins at i, j, k have values α, β, γ. However, if the spins at i, j, k are fixed, then the spin at l is isolated from the rest of the honeycomb lattice. Effectively it sees only a 'magnetic field' $(L_1\alpha + L_2\beta + L_3\gamma)k_BT$, so

$$P(\delta|\alpha, \beta, \gamma) = \tfrac{1}{2}[1 + \delta\tanh(L_1\alpha + L_2\beta + L_3\gamma)]. \tag{4}$$

Since α, β, γ are either $+1$ or -1, equation (4) can be written as

$$P(\delta|\alpha, \beta, \gamma) = \tfrac{1}{2}[1 + \delta(w_1\alpha + w_2\beta + w_3\gamma - w\alpha\beta\gamma)]. \tag{5}$$

The coefficients w, w_1, w_2, w_3 can be obtained from equations (4) and (5) by multiplying by $\alpha\beta\gamma\delta$, $\alpha\delta$, $\beta\delta$, $\gamma\delta$, respectively, and summing over $\alpha, \beta, \gamma, \delta$. Doing this and then using the star-triangle relations (1), one finds after some straightforward algebra that

$$w = \sinh 2K_1 \sinh 2K_2/\sinh 2L_3 \tag{6a}$$

$$w_r/w = \coth 2K_r \qquad r = 1, 2, 3. \tag{6b}$$

Now substitute the expression (5) for $P(\delta|\alpha, \beta, \gamma)$ into equation (3), multiply both sides by $\gamma\delta$, and sum over $\alpha, \beta, \gamma, \delta = \pm 1$. Since $\alpha, \beta, \gamma, \delta$ are the values of the spins σ_i, σ_j, σ_k, σ_l, respectively, it follows that

$$\langle\sigma_k \sigma_l\rangle = w_1\langle\sigma_i \sigma_k\rangle + \sigma w_2\langle\sigma_j \sigma_k\rangle + w_3 - w\langle\sigma_i \sigma_j\rangle. \qquad (7)\dagger$$

This is a linear relation between first and second-neighbour correlations on the honeycomb lattice.

4. Functional equation for the correlations

Now we combine our results (2) and (7). Using also the symmetric analogues of (2), (7) becomes

$$g(L_3, K_3) = w_1 g(K_2, L_2) + L_1 w_2 g(K_1, L_1) + w_3 - w g(K_3, L_3). \qquad (8)$$

We can regard K_1, K_2, K_3 as independent variables, L_1, L_2, L_3 being defined by equation (1). Thus equation (8) is a three-variable relation for the two-variable function $g(K, L)$. We shall show that it determines $g(K, L)$ almost completely.

By negating spins on alternate rows or columns of the square lattice, it is readily found that

$$g(K, L) = -g(-K, L) = g(K, -L) = -g(-K, -L). \qquad (9)$$

Thus it is sufficient to obtain $g(K, L)$ for positive K, L. From now on we therefore consider only the case when K_1, K_2, K_3, L_1, L_2, L_3 are all positive.

Instead of the two-variable function $g(K, L)$, it is convenient to use a function $f(K, k)$, defined in terms of $g(K, L)$ by

$$g(K, L) = \coth 2K \qquad f(K, k), \qquad (10a)$$

$$k = (\sinh 2K \sinh 2L)^{-1}. \qquad (10b)$$

We shall call K the *argument* of $f(K, k)$, and k the *modulus*.

Eliminating L_1 and L_2 between the star-triangle relations (1), one can establish that

$$\cosh 2K_1 \cosh 2K_2 \sinh 2K_3 + \sinh 2K_1 \sinh 2K_2 \cosh 2K_3$$
$$= \sinh 2K_3 \cosh 2L_3. \qquad (11)$$

Also, eliminating K_1 and L_1, or K_2 and L_2, between the relations (1) gives

$$\sinh 2K_1 \sinh 2L_1 = \sinh 2K_2 \sinh 2L_2 = \sinh 2K_3 \sinh 2L_3. \qquad (12)$$

Substituting the expression (10*a*) for g into equation (8), the function f occurs four times. Its argument is different in each case, but from equations (10*b*) and (12) its modulus is the same. Using equations (6) and (11), the resulting equation can be written quite neatly as

$$k^{-1}b \operatorname{sech} 2K_1 \operatorname{sech} 2K_2 \operatorname{sech} 2K_3 = f(K_1, k) + f(K_2, k) + f(K_3, k) - 1 \qquad (13a)$$

† *Note added in proof.* The linear relation (7) between correlations was obtained by Fisher M E 1959 (his equation 81). Very recently, the star-triangle relation has been used to obtain the critical properties of the two-dimensional Ising model via the renormalisation group (Hilhorst H J, Schick M and van Leeuwen J M J 1978).

where

$$b = \coth 2K_3 \coth 2L_3[f(K_3, k) + f(L_3, k) - 1] \tag{13b}$$

and k^{-1} has the common value of the expressions in equation (12), i.e.

$$k^{-1} = \sinh 2K_r \sinh 2L_r \qquad r = 1, 2, 3. \tag{14}$$

Eliminating L_3 between equations (11) and (14) gives an expression for k in terms of the triangular lattice interaction coefficients K_1, K_2, K_3, namely

$$k = \frac{(1-v_1^2)(1-v_2^2)(1-v_3^2)}{4[(1+v_1v_2v_3)(v_1+v_2v_3)(v_2+v_3v_1)(v_3+v_1v_2)]^{1/2}} \tag{15a}$$

where

$$v_r = \tanh K_r \qquad r = 1, 2, 3. \tag{15b}$$

Thus k is the parameter that occurs in previous solutions of the triangular Ising model (Green 1963, Stephenson 1964).

To recapitulate: equation (13) must be true for all positive K_1, K_2, K_3; k is given by equation (15); L_3 by equation (14) or (equivalently) equation (11). We wish to solve equation (13) for the function $f(K, k)$.

5. Solution of the functional equation

From equation (14), L_3 is a function only of K_3 and k. From (13b), b is therefore also a function only of K_3 and k, and this can be exhibited explicitly by writing it as $b(K_3, k)$.

On the other hand, from (13a) it is obvious that b is a symmetric function of K_1, K_2, K_3. Since k is also symmetric, this implies in particular that

$$b(K_2, k) = b(K_3, k). \tag{16}$$

However, for given values of K_3 and k, it is still possible to vary K_2. Thus $b(K, k)$ must be independent of the value of its argument, i.e.

$$b(K, k) = b(k). \tag{17}$$

For given k, b is therefore fixed.

(Strictly, we should consider the allowed ranges of K_2 and K_3. The discussion in § 2 is valid if K_1, K_2, K_3 are all non-negative real. With this restriction, equation (16) implies that equation (17) must hold for all non-negative values of K.)

Now differentiate (13a) along a line in (K_1, K_2, K_3) space on which K_3 and k, and hence L_3, are fixed. The equation (11) can be used to relate the infinitesimal increments in K_1 and K_2, and its symmetric analogues can be used to simplify the result. Doing this, the derivative of equation (13a) can be written

$$a(K_1, k) = a(K_2, k) \tag{18}$$

where, for $r = 1$ or 2,

$$a(K_r, k) = b(k) \tanh^2 2K_r + \tfrac{1}{2} \coth 2L_r f'(K_r, k) \tag{19}$$

$f'(K, k)$ being the derivative of $f(K, k)$ with respect to K.

Just as equation (16) implies equation (17), so does equation (18) imply that $a(K, k)$ is independent of K, i.e.

$$a(K, k) = a(k). \tag{20}$$

Using equations (14), (19) therefore gives a formula for the derivative of f, namely

$$f'(K, k) = 2\frac{a(k) - b(k)\tanh^2 2K}{(1 + k^2 \sinh^2 2K)^{1/2}}. \tag{21}$$

Since the correlation $g(K, K')$ is bounded, the definition (10) implies that $f(0, k) = 0$. Integrating (21) therefore gives, for $0 \leq K < \infty$,

$$f(K, k) = a(k)A(K, k) - b(k)B(K, k) \tag{22a}$$

where

$$A(K, k) = \int_0^{2K} \frac{\mathrm{d}x}{(1 + k^2 \sinh^2 x)^{1/2}} \tag{22b}$$

$$B(K, k) = \int_0^{2K} \frac{\tanh^2 x \, \mathrm{d}x}{(1 + k^2 \sinh^2 x)^{1/2}}. \tag{22c}$$

For given k, f is therefore a linear combination of the functions $A(K, k)$, $B(K, k)$. These functions can be expressed in terms of incomplete elliptic integrals: various such formulae are given in the appendix.

Since $L_3 \to \infty$ when $K_3 \to 0$ for fixed k, from equations (13b) and (17) it follows that $f(\infty, k) = 1$. From (22a) this implies that $a(k)$ and $b(k)$ satisfy the linear relation

$$a(k)A(\infty, k) - b(k)B(\infty, k) = 1. \tag{23}$$

Using this, one can verify that the functional relation (13) is satisfied by the solution (22), so we have extracted as much information from it as possible. It only remains to calculate either of the single-variable functions $a(k)$ or $b(k)$.

6. Differential equations for $a(k)$, $b(k)$

To determine $a(k)$ and $b(k)$, we note that

$$g(K, L) = -\frac{\partial \psi(K, L)}{\partial K} \tag{24}$$

where $k_B T\psi$ is the free energy per site of a square lattice Ising model with interaction coefficients K and L for horizontal and vertical edges, respectively. This $\psi(K, L)$ must be a symmetric function of K and L. Differentiating equation (24) with respect to L and using the definition (10), it follows that the function $f(K, k)$ must satisfy the symmetry relation

$$\left(\frac{\partial f(K, k)}{\partial k}\right)_K = \left(\frac{\partial f(L, k)}{\partial k}\right)_L. \tag{25}$$

The $f(L, k)$ on the RHS of equation (25) can be expressed in terms of $f(K, k)$ by using (13b), with the suffix 3 deleted. Doing this, and using (10b), (17) and (21), the relation

(25) becomes

$$k\frac{\partial f(K,k)}{\partial k}=\tfrac{1}{2}[a(k)-b(k)+kb'(k)]C(K,k) \tag{26}$$

where

$$C(K,k)=\frac{\tanh 2K}{(1+k^2\sinh^2 2K)^{1/2}}. \tag{27}$$

On the LHS of equation (26), and from now on, differentiation with respect to k is to be understood as being performed for fixed K; $a'(k)$ and $b'(k)$ are the derivatives of $a(k)$ and $b(k)$.

The functions $A(K, k)$ and $B(K, k)$ defined by (22b) and (22c) are analytic for all positive K and k. Setting

$$k'^2=1-k^2 \tag{28}$$

they satisfy the following differential equations:

$$k\frac{\partial}{\partial k}A(K,k)=B(K,k)-A(K,k)+C(K,k) \tag{29a}$$

$$k\frac{\partial}{\partial k}[k'^2B(K,k)]=k'^2B(K,k)-A(K,k)+C(K,k). \tag{29b}$$

(These identities can readily be verified by differentiation with respect to K.)

Substituting the expression (22a) for $f(K, k)$ into (26) and using (29), (26) becomes

$$[-kb'+a-(1+k^2)c][B(K,k)+\tfrac{1}{2}C(K,k)]+(ka'-a+c)A(K,k)=0 \tag{30}$$

where

$$c\equiv c(k)=b(k)/k'^2. \tag{31}$$

For given k, the functions $A(K, k)$, $B(K, k)$, $C(K, k)$ are linearly independent, so the relation (30) implies that

$$ka'=a-c \tag{32a}$$

$$kb'=a-(1+k^2)c. \tag{32b}$$

Eliminating a and b between (31) and (32) gives

$$\frac{\mathrm{d}}{\mathrm{d}k}\left(kk'^2\frac{\mathrm{d}c}{\mathrm{d}k}\right)-kc=0. \tag{33}$$

This is a second-order homogeneous linear differential equation for $c(k)$. Once $c(k)$ is known, $b(k)$ and $a(k)$ can be obtained from equations (31) and (32b).

7. Final determination of the correlations

The above equations apply to all positive values of k. However, the differential equation (33) is singular at $k=1$, so we must consider the cases $k<1$ and $k>1$ separately.

7.1. Low-temperature case: $k < 1$

Let $\mathscr{K}(k)$, $E(k)$ be the complete elliptic integrals of the first and second kinds, of modulus k (Gradshteyn and Ryzhik 1965, hereinafter referred to as GR, § 8.112). For $k < 1$ both $\mathscr{K}(k)$ and $\mathscr{K}(k')$ are solutions of (33) (GR § 8.124.1), so the general solution is

$$c(k) = \lambda \mathscr{K}(k) + \mu \mathscr{K}(k'), \tag{34}$$

where λ, μ are constants.

As $k \to 0$, $\mathscr{K}(k) \to \pi/2$ while $\mathscr{K}(k') \to \infty$. If $\mu \neq 0$, it follows from equations (31), (32), and (22) that $a(k)$, $b(k)$ and $f(K, k)$ become infinite (for fixed K). However, from equations (2) and (10), $|f(K, k)| < 1$, so f is bounded. This means that μ must be zero.

From equations (31), (32) and GR § 8.123.2, it follows that

$$a(k) = \lambda E(k) \qquad b(k) = \lambda k'^2 \mathscr{K}(k). \tag{35}$$

Changing the integration variable x in (22) to α, where $\tan\alpha = \sinh x$, it is found that

$$A(\infty, k) = \mathscr{K}(k'), B(\infty, k) = [\mathscr{K}(k') - E(k')]/k'^2. \tag{36}$$

The constant λ can now be evaluated by substituting the expressions (35) and (36) into the condition (23). Using the identity (GR § 8.122)

$$E(k)\mathscr{K}(k') + E(k')\mathscr{K}(k) - \mathscr{K}(k)\mathscr{K}(k') = \tfrac{1}{2}\pi, \tag{37}$$

we find that

$$\lambda = 2/\pi. \tag{38}$$

7.2. High-temperature case: $k > 1$

In this case the general solution of equation (33) is

$$c(k) = l[\lambda \mathscr{K}(l) + \mu \mathscr{K}(l')] \tag{39}$$

where λ, μ are constants (not necessarily the same as those in equation (34)), and

$$l = k^{-1} \qquad l' = (1 - l^2)^{1/2}. \tag{40}$$

As $k \to \infty$, $\mathscr{K}(l) \to \pi/2$ and $\mathscr{K}(l') \to \infty$. If $\mu \neq 0$ it follows from equations (31), (32) and (22) that $f(K, k) \to \infty$ for fixed K. Since this is not allowed, μ must again be zero. Equations (31) and (32) then give

$$\begin{aligned} a(k) &= \lambda [E(l) - l'^2 \mathscr{K}(l)]/l \\ b(k) &= -\lambda l'^2 \mathscr{K}(l)/l. \end{aligned} \tag{41}$$

Changing the integration variable in (22) to α, where $\tan\alpha = k \sinh x$, gives

$$\begin{aligned} A(\infty, k) &= l\mathscr{K}(l') \\ B(\infty, k) &= l[E(l') - l^2 \mathscr{K}(l')]/l'^2. \end{aligned} \tag{42}$$

Substituting (41) and (42) into (23) and using the identity (37) with k replaced by l, we again find that $\lambda = 2/\pi$. Thus λ, μ in equation (39) do in fact have the same values as in equation (34).

8. Summary

The elliptic integrals of moduli k, k^{-1} are related by a Landen transformation (GR § 8.126) to those of modulus

$$k_1 = 2k^{1/2}/(1+k). \tag{43}$$

Using this, the above results for $a(k)$, $b(k)$ can be written in the single form, true for all positive k,

$$\begin{aligned} a(k) &= [(1+k)E(k_1)+(1-k)\mathscr{K}(k_1)]/\pi \\ b(k) &= 2(1-k)\mathscr{K}(k_1)/\pi. \end{aligned} \tag{44}$$

Together with equations (2), (10) and (22), this gives the nearest-neighbour correlations of the honeycomb, triangular and square Ising models.

One interesting feature of this derivation is that it makes it very clear how the non-analyticity at the critical point $k=1$ occurs. Eliminating $a(k)$ between equations (22a) and (23) gives

$$f(K,k) = \frac{A(K,k)}{A(\infty,k)} - b(k)\left(B(K,k) - \frac{B(\infty,k)A(K,k)}{A(\infty,k)}\right). \tag{45}$$

All the functions and ratios on the RHS are analytic at $k=1$, except for the single 'coefficient' $b(k)$. This is independent of K, so all honeycomb, triangular and square Ising models have the same singularity in their internal energy, namely that of $b(k)$. At $k=1$, $b(k)$ is continuous but non-analytic, being given near $k=1$, by

$$b(k) \simeq \pi^{-1}(1-k^2)\ln[16/|1-k^2|]. \tag{46}$$

The symmetric logarithmic divergence of the specific heat follows from equation (46).

The result (22) and (44) for $f(K,k)$ can be written very neatly in terms of elliptic functions. This is done in the appendix. We have verified that the result is the same as that of Onsager (1944, equations (113a) and (A5.1)).

Appendix

Define K^*, θ, ϕ, Δ by

$$\sinh 2K^* = k\tan\theta = \tan\phi = k\sinh 2K \tag{A1}$$

$$\Delta = \tanh 2K(1+k^2\sinh^2 2K)^{1/2}. \tag{A2}$$

Let F, E be the incomplete elliptic integrals (GR § 8.111). Then, using equations (22), (27), (28) and (40),

$$A(K,k) = -\mathrm{i}F(2\mathrm{i}K,k) = -\mathrm{i}lF(2\mathrm{i}K^*,l) = F(\theta,k') = lF(\phi,l') \tag{A3}$$

$$\begin{aligned} k'^2B(K,k) &= -\Delta - \mathrm{i}E(2\mathrm{i}K,k) \\ &= -\Delta - \mathrm{i}[E(2\mathrm{i}K^*,l) - l'^2F(2\mathrm{i}K^*,l)]/l \\ &= F(\theta,k') - E(\theta,k') \\ &= -k'^2C(K,k) - [E(\phi,l') - l^2F(\phi,l')]/l. \end{aligned} \tag{A4}$$

The function $f(K, k)$ can be expressed in terms of the elliptic theta functions H_1, Θ_1 and their derivatives H_1', Θ_1' (GR §§ 8.191 and 8.192):

$$k<1: -\mathrm{i}\, \mathrm{sn}(\mathrm{i}a, k)=\sinh 2K, 0<a<\mathscr{K}(k')$$

$$f(K, k)=\frac{2\mathrm{i}\mathscr{K}(k)}{\pi}\frac{H_1'(\mathrm{i}a, k)}{H_1(\mathrm{i}a, k)} \tag{A5a}$$

$$k>1: -\mathrm{i}\, l\, \mathrm{sn}(\mathrm{i}a, l)=\sinh 2K, 0<a<\mathscr{K}(l')$$

$$f(K, k)=\frac{2\mathrm{i}\mathscr{K}(l)}{\pi}\frac{\Theta_1'(\mathrm{i}a, l)}{\Theta_1(\mathrm{i}a, l)}. \tag{A5b}$$

This parameter a is *not* that used in the text: it is that used by Onsager (1944, A2.2).

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